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COMMENT

Asymptotic approximations for certain 6-j and 9-j symbols

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Abstract. An error in the phase factor of a recent asymptotic approximation by Chen *et al* (1999 *J. Phys. A: Math. Gen.* **32** 537–53) for certain 6-j symbols in terms of 3-j symbols is corrected. A related asymptotic approximation for 9-j symbols with large angular momenta in the four 'corners' is pointed out.

In a recent paper, Chen *et al* [1] gave a justification of asymptotic approximations for 6-j symbols and Racah polynomials originally published by Ponzano and Regge [2]. These approximations are useful in a variety of systems, such as the Rydberg states of molecules [3], where the rotational angular momentum is often large compared with the other angular momenta, giving Hund's angular momentum coupling cases d or e.

The purpose of this comment is to point out that the phase given in [1] for the asymptotic 6-j coefficient

$$\begin{Bmatrix} a & b & e \\ d+R & c+R & f+R \end{Bmatrix} \qquad (a,\ldots,f) = \mathrm{O}(1) \qquad R \sim \infty$$

is incorrect for certain cases involving half-integer quantum numbers.

To correct the phase we can start by restoring the factor $(-)^{2R}$ omitted in equations (2.13) and (2.14) of [1]. If we make the replacement $s \to s + \alpha_e$ in equation (2.14), the sum S_1 becomes the same as that in equation (1.4) for the 3-j symbol

$$\begin{pmatrix} a & b & e \\ c-f & f-d & d-c \end{pmatrix}.$$

The phase factor in the relation between the 6-j and 3-j symbols is then found to be

$$(-)^{\alpha_e+2R-a+b+d-c} = (-)^{a+b+e+2a+2d+2R} = (-)^{a+b+e+2(d+c+f+3R)}. \tag{1}$$

We thus obtain

$$\begin{Bmatrix} a & b & e \\ d+R & c+R & f+R \end{Bmatrix} \sim \frac{(-)^{a+b+e+2(d+c+f+3R)}}{(2R)^{1/2}} \begin{pmatrix} a & b & e \\ c-f & f-d & d-c \end{pmatrix}. \tag{2}$$

This result reproduces the original phase factor given by Ponzano and Regge [2], with the generalization that here R can be an integer or half-integer.

The factor $(-)^{a+b+e}$ can be absorbed by changing the sign of the components in the 3-j symbol. With a change of notation, the result can be rewritten as [3]

$$\begin{Bmatrix} a & b & c \\ D & E & F \end{Bmatrix} \sim \frac{(-)^{2(D+E+F)}}{(2R+1)^{1/2}} \begin{pmatrix} a & b & c \\ F-E & D-F & E-D \end{pmatrix} \tag{3}$$

where a, b, c are small, D, E, F are large, and $R \approx \frac{1}{3}(D+E+F)$. This result is equivalent to that given by Alder *et al* [4]. Relative to this formula, that of [1] lacks the factor $(-)^{2(D+E+F)}$. The necessity for this factor can be seen from a particular case, the (exact) relation

$$\begin{Bmatrix} a & 0 & a \\ D & E & D \end{Bmatrix} = \frac{(-)^{2E}}{(2D+1)^{1/2}} \begin{pmatrix} a & 0 & a \\ D-E & 0 & E-D \end{pmatrix} \tag{4}$$

where the factor $(-)^{2E}$ is -1 for half-integer E, which occurs for half-integer $(D-a)$.

An interesting asymptotic relation for certain 9-j symbols can be derived from the 6-j result (3), namely [3]

$$\begin{Bmatrix} A & b & C \\ d & e & f \\ G & h & I \end{Bmatrix} \sim \frac{(-)^{b+C-d-G}}{(2R+1)} \begin{pmatrix} b & e & h \\ C-A & \epsilon & G-I \end{pmatrix} \begin{pmatrix} d & e & f \\ G-A & \epsilon & C-I \end{pmatrix} \tag{5}$$

where the 'corner' angular momenta are large, the others are small, $R \approx \frac{1}{4}(A+C+G+I)$ and $\epsilon = A - C - G + I$. This approximation is obtained by using equation (3) in the second-last equation of page 143 of [5] and then summing by means of the third equation of page 141 of [5]. This asymptotic approximation has been tested by numerical calculations for a range of values of the quantum numbers.

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